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$SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$, the classical and quantum q-deformations of the SU(2) algebra: IV. Geometric quantization and Hopf algebraic structures†

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Abstract. In the q-deformed oscillator system, the Hopf algebraic structures of $SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$ are constructed in the scheme of geometric quantization. The classical and quantum representation spaces are analysed with respect to whether or not q is a root of unity.

1. Introduction

The quantum groups [1–10], which are deeply rooted in many physical systems and theories, are q-deformations of Lie algebras with non-trivial Hopf algebraic structures. It is a common opinion that the deformation parameter q is connected with the Planck constant, i.e. $\ln q \propto \hbar$. According to this statement, when $\hbar \to 0$, $q \to 1$, and the quantum systems reduce to classical ones, while the q-deformed symmetries are changed to Lie symmetries upon canonical quantization. As we stressed in [11–13], this is not the case in principle. In fact, the q-deformation and the $\hbar$-quantization can be two independent processes. It is possible to find classical systems possessing q-deformed symmetries, and these systems become quantum ones with quantum group symmetries after quantization. Inversely, when we take $\hbar \to 0$ in the systems with quantum group symmetries, the systems become classical with q-deformed symmetries.

This idea makes it meaningful to look for the validity of the quantum groups as dynamical symmetries in physical systems with violations of the perfect Lie symmetries. As q is near unity, there are exact symmetries in view of quantum group theory, but small violations of the Lie symmetries are allowed, though the quantum symmetries certainly embrace the more general case where q is not near unity. A few attempts have been made in this direction in molecular physics and the problem of heavy ion resonances [14–16] where the violations of the Lie algebra SU(2) and simple harmonic oscillation are well known.

In the q-oscillator approach, we have shown that the q-deformed algebra $SU_q(2)$ can be realized both at classical and quantum levels, and the classical q-deformed

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algebra $SU_{q,\hbar\to 0}(2)$ is obtained by deforming the classical observables. However, an important problem that remains to be solved is whether for such a classical deformed algebra there still exists a non-trivial Hopf algebraic structure like the usual quantum algebras. Obviously we should solve this problem to support our point of view. The main purpose of this paper is to construct the Hopf algebraic structure for the classical deformed algebra, thus confirming our previous works [11–13] on classical and quantum q-deformations of the SU(2) algebra.

Recently, two of us (SMF and HYG) [17, 18] have solved this problem in terms of the geometric quantization [19] method by deforming the symplectic structure in addition to the classical observables on a 2-sphere. We will follow the key point of this method to construct the Hopf algebraic structure for the classical q-deformed algebra $SU_{q,\hbar\to 0}(2)$ in the q-oscillator approach.

The difficulty in constructing the Hopf algebra for the classical q-deformed algebra is that the q-deformed algebra $SU_{q,\hbar\to 0}(2)$ in Poisson brackets is generated by classical observables which form an associative but commutative algebra under free multiplication between the functions. In order to construct the Hopf algebraic structure for the classical q-deformed algebra, we have to find a set of non-commutative operators, such that they not only form an associative and non-commutative algebra (with the unit) under free multiplication but also generate the classical q-deformed algebra $SU_{q,\hbar\to 0}(2)$ in Lie brackets rather than in Poisson brackets. Once such operators are given, it is easy to set up the Hopf algebraic structure. As pointed out in [17, 18], in view of the geometric quantization method, such operators defined on the line bundle over the classical phase space of the harmonic oscillator system are nothing but the prequantization operators of the q-deformed observables that form the classical q-deformed algebra $SU_{q,\hbar\to 0}(2)$ in Poisson brackets. Since the prequantization line bundle over the phase space is still classical, there should be no room for the Planck constant $\hbar$ in the set of non-commutative operators. This is a subtle but crucial difference from the usual expression of the prequantization operators in the literature (see [19] and references therein). Physically, the Planck constant $\hbar$ is completely quantum characteristic. It should appear only after suitable polarization is taken according to geometric quantization. In other words, polarization turns the classical q-deformed algebra $SU_{q,\hbar\to 0}(2)$ into the quantum q-deformed algebra $SU_{q,\hbar}(2)$ for which the Hopf algebraic structure is well known.

We also present the classical and quantum representations of the classical q-deformed algebra $SU_{q,\hbar\to 0}(2)$ and the quantum q-deformed algebra $SU_{q,\hbar}(2)$, respectively, with respect to whether q is a root of unity or not. After the polarization, the classical representations reduce to the quantum ones, as they should.

The paper is organized as follows. In §2, we investigate the Hopf algebraic structures of $SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$ algebras. Section 3 is devoted to the representations of the above algebras, and some brief remarks and discussions are given in the last section.

2. Geometric quantization and Hopf algebraic structures of $SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$ algebras

In order to construct the Hopf algebraic structures of the classical q-deformed algebra $SU_q(2)$, we first construct a set of non-commutative operators such that they form the $SU_{q,\hbar\to 0}(2)$ algebra in Lie brackets. As analysed in the previous section, this can be reached at the prequantization level by geometric quantization.

It has been pointed out that in a classical mechanical oscillator system with Hamiltonian and symplectic form

$$H = \sum_{i=1}^{2} \bar{z}_i z_i$$

$$\Omega = -\mathrm{i} \sum_{i=1}^{2} \mathrm{d}\bar{z}_i \wedge \mathrm{d}z_i \tag{1}$$

where $z_i = (p_i + \mathrm{i}q_i)/\sqrt{2}$, $\bar{z}_i = (p_i - \mathrm{i}q_i)/\sqrt{2}$, and the classical quantities $J_+ = z_1\bar{z}_2$, $J_- = z_2\bar{z}_1$ and $J_3 = \frac{1}{2}(z_1\bar{z}_1 - z_2\bar{z}_2)$ constitute the SU(2) algebra in Poisson brackets, i.e.

$$[J_+, J_-]_{\mathrm{PB}} = -\mathrm{i}2J_3 \qquad [J_3, J_\pm]_{\mathrm{PB}} = -\mathrm{i}(\pm J_\pm). \tag{2}$$

The q-deformed quantities

$$J'_+ = z_1\bar{z}'_2 \qquad J'_- = z_2\bar{z}'_1 \qquad J'_3 = J_3 \tag{3}$$

generate the $\mathrm{SU}_{q,\hbar\to 0}(2)$ algebra with the following Poisson relations:

$$[J'_+, J'_-]_{\mathrm{PB}} = -\mathrm{i}[2J'_3]_q \qquad [J'_3, J'_\pm]_{\mathrm{PB}} = -\mathrm{i}(\pm J'_\pm) \tag{4}$$

where the well-known relations

$$z'_i = \frac{1}{\sqrt{\gamma \sinh \gamma}} \frac{\sinh(\gamma z_i \bar{z}_i)}{z_i \bar{z}_i} z_i \qquad \bar{z}'_i = \frac{1}{\sqrt{\gamma \sinh \gamma}} \frac{\sinh(\gamma z_i \bar{z}_i)}{z_i \bar{z}_i} \bar{z}_i \tag{5}$$

and

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$$

have been applied.

The prequantization line bundle over the phase space of the undeformed oscillators [17] is of curvature Ω and connection (symplectic 1-form)

$$\theta = -\mathrm{i}(\bar{z}_1 \mathrm{d}z_1 + \bar{z}_2 \mathrm{d}z_2). \tag{6}$$

In terms of the geometric quantization method, the prequantization operators of the observable f on the phase space V (represented by z_i and $\bar{z}_i$) can be obtained by the following prequantization map:

$$f \longrightarrow \tilde{f} = -\mathrm{i}(X_f - \mathrm{i}\theta(X_f)) + f \tag{7}$$

where X_f is the Hamiltonian vector field of f. For the reasons stated in §1, here we do not insert the virtually irrelevant Planck constant.

From formula (1) we have

$$X_{\bar{z}_i} = \mathrm{i}\frac{\partial}{\partial z_i} \qquad X_{z_i} = -\mathrm{i}\frac{\partial}{\partial \bar{z}_i} \qquad i = 1, 2. \tag{8}$$

Hence using (7) the prequantization versions of z_i and $\bar{z}_i$ are, respectively,

$$\omega_i = -\frac{\partial}{\partial \bar{z}_i} + z_i \qquad \overline{\omega}_i = \frac{\partial}{\partial z_i} \tag{9}$$

with the following commutation relations:

$$[\omega_i, \overline{\omega}_j] = -\delta_{ij}. \tag{10}$$

To obtain the prequantization operators of $J'_\pm$ and J'_3, we need a suitable ordering of ω_i and $\bar{\omega}_i$. From formulae (2) and (4) we have

$$\begin{aligned} \tilde{J}'_+ &= \frac{1}{\sqrt{\gamma \sinh \gamma}} \frac{\omega_1}{\omega_2} \sinh(\gamma \omega_2 \bar{\omega}_2) \\ \tilde{J}'_- &= \frac{1}{\sqrt{\gamma \sinh \gamma}} \frac{\omega_2}{\omega_1} \sinh(\gamma \omega_1 \bar{\omega}_1) \\ \tilde{J}'_3 &= \tfrac{1}{2}(\omega_1 \bar{\omega}_1 - \omega_2 \bar{\omega}_2). \end{aligned} \tag{11}$$

It is straightforward to find their commutation relations by (10),

$$[\tilde{J}'_+, \tilde{J}'_-] = \frac{1}{\gamma} \sinh\left(2\gamma \tilde{J}'_3\right) \qquad [\tilde{J}'_3, \tilde{J}'_\pm] = \pm \tilde{J}'_\pm. \tag{12}$$

Algebra (12) is isomorphic to the Poisson algebra (5). Furthermore, as the polarization has not been introduced, the square integrable section space (i.e. the Hilbert space) is still classical. That is, the operators $\tilde{J}'_\pm$ and $\tilde{J}'_3$ are still classical quantities and the algebras generated by them are only a classical realization of the q-deformed algebra in Lie brackets rather than in Poisson brackets.

With formulae (11) and (12) we are ready to define the co-product Δ, co-unit ϵ and antipodal mapping S,

$$\begin{aligned} &\Delta(\tilde{J}'_3) = \tilde{J}'_3 \otimes 1 + 1 \otimes \tilde{J}'_3 \\ &\Delta(\tilde{J}'_\pm) = \tilde{J}'_\pm \otimes q^{\tilde{J}'_3} + q^{-\tilde{J}'_3} \otimes \tilde{J}'_3 \\ &S(\tilde{J}'_3) = -\tilde{J}'_3 \qquad S(\tilde{J}_3) = -q^{\pm 1} \tilde{J}'_\pm \\ &\epsilon(\tilde{J}'_\pm) = \epsilon(\tilde{J}'_3) = 0 \qquad \epsilon(1) = 1. \end{aligned} \tag{13}$$

These three Hopf operations are algebra homomorphisms and anti-homomorphism, i.e.

$$\begin{aligned} &\Delta : A \rightarrow A \otimes A \qquad \Delta(ab) = \Delta(a)\Delta(b) \\ &S : A \rightarrow A \qquad S(ab) = S(b)S(a) \\ &\epsilon : A \rightarrow C \qquad \epsilon(ab) = \epsilon(a)\epsilon(b) \end{aligned} \tag{14}$$

where a, b are elements of algebra A (which is $\mathrm{SU}_{q\hbar\to 0}(2)$ at present), and C is the field of complex numbers. The above three operations, supplemented by the identical mapping id and multiplication m, are consistent, i.e.

$$
\begin{aligned}
&(\mathrm{id}\otimes\Delta)\Delta(a) = (\Delta\otimes\mathrm{id})\Delta(a)\\
&\mathrm{m}(\mathrm{id}\otimes S)\Delta(a) = \mathrm{m}(S\otimes\mathrm{id})\Delta(a) = \epsilon(a)\cdot 1\\
&(\epsilon\otimes\mathrm{id})\Delta(a) = (\mathrm{id}\otimes\epsilon)\Delta(a) = a
\end{aligned}
\tag{15}
$$

and compatible with the algebraic relations (12).

Formulae (13) define neither commutative nor co-commutative Hopf algebras. Thus it is verified that the classical q-deformed algebra $\mathrm{SU}_{q,\hbar\to 0}(2)$ does possess non-trivial Hopf algebraic structure as the usual quantum groups do, though it has nothing to do with quantization at all. In fact, at the prequantization level, the uncertainty relation is not satisfied since the square integrable sections of the prequantization line bundle over the classical phase space can have arbitrarily small support. This is why the prequantization operators $\tilde{J}'_\pm$ and $\tilde{J}'_3$ remain at the classical level.

In the geometric quantization approach, to quantize the system on the symplectic space (V,Ω) introduces a polarization. Let us take the basis of the polarization to be $X = \{\partial/\partial\bar{z}_1, \partial/\partial\bar{z}_2\}$ (Kähler polarization). Then from the quantum map [17], we obtain the quantum version of a given classical observable f if it preserves the polarization

$$
f \longrightarrow \hat{f} = -\mathrm{i}\hbar(X_f - \mathrm{i}\theta(X_f)) + f - \tfrac{1}{2}\mathrm{i}\hbar a \tag{16}
$$

where a is determined by the formula

$$
[X_f, X] = aX. \tag{17}
$$

Therefore we obtain the quantum operators of z_i and $\bar{z}_i$,

$$
\hat{\omega}_i = z_i \qquad \hat{\bar{\omega}}_i = \hbar\frac{\partial}{\partial z_i} \tag{18}
$$

with commutation relation

$$
[\hat{\omega}_i, \hat{\bar{\omega}}_j] = -\hbar\delta_{ij}. \tag{19}
$$

Similarly, the quantum versions of $J'_\pm$ and J'_3 are

$$
\begin{aligned}
\hat{J}'_+ &= \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{z_1}{z_2}\sinh\left(\gamma\hbar z_2\frac{\partial}{\partial z_2}\right)\\
\hat{J}'_- &= \frac{1}{\sqrt{\gamma\sinh\gamma}}\frac{z_2}{z_1}\sinh\left(\gamma\hbar z_1\frac{\partial}{\partial z_1}\right)\\
\hat{J}'_3 &= -\frac{\hbar}{2}\left(z_2\frac{\partial}{\partial z_2} - z_1\frac{\partial}{\partial z_1}\right).
\end{aligned}
\tag{20}
$$

In terms of (19), $\hat{J}'_{\pm}$ and $\hat{J}'_3$ give rise to the quantum q-deformed algebra $\mathrm{SU}_{q,\hbar}(2)$,

$$[\hat{J}'_+, \hat{J}'_-] = \frac{\sinh \gamma\hbar}{\gamma} \frac{\sinh\left(2\gamma \hat{J}'_3\right)}{\sinh \gamma} \qquad [\hat{J}'_3, \hat{J}'_{\pm}] = \hbar \hat{J}'_{\pm} \tag{21}$$

which is isomorphic to the usual quantum algebra of SU(2), but with an additional parameter of $\hbar$ characterizing the physical quantization of the system.

The Hopf algebraic structure of $\mathrm{SU}_{q,\hbar}(2)$ is identical to that of $\mathrm{SU}_{q,\hbar\to 0}(2)$ in form,

$$\begin{aligned}
&\Delta(\hat{J}'_3) = \hat{J}'_3 \otimes 1 + 1 \otimes \hat{J}'_3 \\
&\Delta(\hat{J}'_{\pm}) = \hat{J}'_{\pm} \otimes q^{\hat{J}'_3} + q^{-\hat{J}'_3} \otimes \hat{J}'_3 \\
&S(\hat{J}'_3) = -\hat{J}_3{}' \qquad S(\hat{J}_3) = -q^{\pm 1} \hat{J}'_{\pm} \\
&\epsilon(\hat{J}'_{\pm}) = \epsilon(\hat{J}'_3) = 0 \qquad \epsilon(1) = 1.
\end{aligned} \tag{22}$$

It is easily checked that they are algebra (anti-)homomorphisms similar to those in (14) and satisfy the consistency conditions in (15).

3. The representations of the algebras $\mathrm{SU}_{q,\hbar\to 0}(2)$ and $\mathrm{SU}_{q,\hbar}(2)$

In this section, we first give the representations of $\mathrm{SU}_{q,\hbar\to 0}(2)$ in the prequantization line bundle. After quantization, due to the polarization introduced in §2, the classical representation space reduces to the quantum Hilbert space which consists of holomorphic sections in the Kähler polarization. In other words, they are covariantly constant along the polarization.

Consider the representation of the algebra spanned by $\{\omega, \bar{\omega}, \omega\bar{\omega}\}$ with

$$\begin{aligned}
\omega &= z + \frac{\partial}{\partial \bar{z}} \\
\bar{\omega} &= -\frac{\partial}{\partial z}.
\end{aligned} \tag{23}$$

It is easy to see that the representation space is

$$F = \{f_n^{\alpha}, \quad n = 0, 1, 2, \ldots\} \tag{24}$$

where

$$f_n^{\alpha} = \frac{(z+\alpha)^n}{\sqrt{n!}} e^{\alpha \bar{z}} \tag{25}$$

and α is an arbitrary constant in C. It is easy to find that

$$\begin{aligned}
\omega f_n^{\alpha} &= \sqrt{n+1} f_{n+1}^{\alpha} \\
\bar{\omega} f_n^{\alpha} &= -\sqrt{n} f_{n-1}^{\alpha} \\
\omega\bar{\omega} f_n^{\alpha} &= n f_n^{\alpha}.
\end{aligned} \tag{26}$$

All the states f_n^α can be connected by the ladder operators ω and $\bar{\omega}$, i.e.

$$0 \underset{\bar{\omega}}{\overset{\omega}{\rightleftarrows}} 1 \underset{\bar{\omega}}{\overset{\omega}{\rightleftarrows}} 2 \underset{\bar{\omega}}{\overset{\omega}{\rightleftarrows}} \cdots \underset{\bar{\omega}}{\overset{\omega}{\rightleftarrows}} n \underset{\bar{\omega}}{\overset{\omega}{\rightleftarrows}} (n+1) \underset{\bar{\omega}}{\overset{\omega}{\rightleftarrows}} \cdots . \tag{27}$$

Now we consider the q-deformation of the above algebra. The deformed generators are ω_q, $\bar{\omega}_q$, and $\omega\bar{\omega}$ defined by

$$\begin{aligned} \omega_q &= \omega = z + \frac{\partial}{\partial \bar{z}} \\ \bar{\omega}_q &= \frac{1}{\omega}[\omega\bar{\omega}]_q = \bar{\omega}\frac{[\omega\bar{\omega}]_q}{\omega\bar{\omega}} \end{aligned} \tag{28}$$

which satisfy the following algebraic relations:

$$\begin{aligned} [\omega_q, \bar{\omega}_q] &= [\omega\bar{\omega}]_q - [\omega\bar{\omega} - 1]_q \\ [\omega\bar{\omega}, \omega_q] &= -\omega_q \\ [\omega\bar{\omega}, \bar{\omega}_q] &= \bar{\omega}_q . \end{aligned} \tag{29}$$

The representation space is

$$G = \{g_n^\alpha, \quad n = 0, 1, 2, \ldots, \alpha \in \mathbf{R}\} \tag{30}$$

where

$$g_n^\alpha = \frac{(z+\alpha)^n}{[n]_q!} e^{\alpha \bar{z}} = \frac{\omega_q^n}{[n]_q!} e^{\alpha \bar{z}} . \tag{31}$$

The actions of the operators on this space give rise to

$$\omega_q, \ \bar{\omega}_q, \ \omega\bar{\omega} : \quad G \longrightarrow G \tag{32}$$

or more explicitly

$$\begin{aligned} \omega_q g_n^\alpha &= [n+1]_q g_{n+1}^\alpha \\ \bar{\omega}_q g_n^\alpha &= g_{n-1}^\alpha \\ \omega\bar{\omega} g_n^\alpha &= n g_n^\alpha . \end{aligned} \tag{33}$$

The algebraic relations in (29), the classical Hilbert space in (30) and the actions of the generators in the Hilbert space in (33) are valid for arbitrary complex values of q. However, the characteristics of the representation space are different depending on whether q is a root of unity or not.

3.1. q not a root of unity

The properties of the representation space of the deformed algebra are identical to those of the algebra without deformation. The essential property is that the states in the Hilbert space can all be raised or lowered by the ladder operator ω_q and $\bar{\omega}_q$. Explicitly,

$$0 \underset{\bar{\omega}_q}{\overset{\omega_q}{\rightleftarrows}} 1 \underset{\bar{\omega}_q}{\overset{\omega_q}{\rightleftarrows}} 2 \underset{\bar{\omega}_q}{\overset{\omega_q}{\rightleftarrows}} \cdots \underset{\bar{\omega}}{\overset{\omega}{\rightleftarrows}} n \underset{\bar{\omega}_q}{\overset{\omega_q}{\rightleftarrows}} (n+1) \underset{\bar{\omega}_q}{\overset{\omega_q}{\rightleftarrows}} \cdots . \tag{34}$$

3.2. *q a root of unity*

Some odd properties appear in this case. Suppose that q is the least positive integer satisfying $q^p = \pm 1$, then $[p]_q = 0$. Therefore the representation space splits into infinite indecomposable invariant subspaces as shown in the following chain of states:

$$0 \underset{\bar{\omega}_q}{\overset{\omega_q}{\rightleftarrows}} 1 \underset{\bar{\omega}_q}{\overset{\omega_q}{\rightleftarrows}} 2 \underset{\bar{\omega}_q}{\overset{\omega_q}{\rightleftarrows}} \cdots \overset{\bar{\omega}_q}{\longleftarrow} p \underset{\bar{\omega}}{\overset{\omega}{\rightleftarrows}} (p-1) \overset{\bar{\omega}_q}{\longleftarrow} p \underset{\bar{\omega}_q}{\overset{\omega_q}{\rightleftarrows}} (p+1) \underset{\bar{\omega}_q}{\overset{\omega_q}{\rightleftarrows}} \cdots . \tag{35}$$

This is due to the fact that, when we act on the states,

$$\omega_q^n = 0 \qquad (n \geqslant p) \tag{36}$$

and therefore

$$\omega_q^n g_{p-1}^\alpha = 0 \qquad \omega_q^n g_{np-1}^\alpha = 0 \qquad (n = 1, 2, \ldots). \tag{37}$$

Although there are such unusual relations, the Lusztig-like operator

$$L_q = \frac{\omega_q^p}{[p]_q!} \tag{38}$$

is still well defined regardless of the nilpotencies of $[p]_q$ and ω_q^p. Therefore the states defined in (31) avoid possible nilpotencies.

It should be mentioned that the $\mathrm{SU}_q(2)$ algebra at the prequantization level can be realized by the above prequantized differential operators as follows:

$$\begin{aligned} \tilde{J}'_+ &= \sqrt{\frac{\sinh\gamma}{\gamma}}\omega_{1q}\bar{\omega}_{2q} \\ \tilde{J}'_- &= \sqrt{\frac{\sinh\gamma}{\gamma}}\omega_{2q}\bar{\omega}_{1q} \\ \tilde{J}'_3 &= \frac{1}{2}(\omega_1\bar{\omega}_1 - \omega_2\bar{\omega}_2). \end{aligned} \tag{39}$$

It is not difficult to check that the $\mathrm{SU}_q(2)$ algebraic relations hold:

$$\begin{aligned} [\tilde{J}'_+, \tilde{J}'_-] &= \frac{\sinh\gamma}{\gamma}[2\tilde{J}'_3]_q \\ [\tilde{J}'_3, \tilde{J}'_\pm] &= \pm\tilde{J}'_\pm. \end{aligned} \tag{40}$$

Now we are in a position to consider the representations of the $\mathrm{SU}_q(2)$ symmetry in the prequantized system. The representation spaces should be the tensor products of the Hilbert spaces of the algebras generated by $\{\omega_{iq}, \bar{\omega_{iq}}, \omega_i\bar{\omega}_i\}$ with $i = 1, 2$. Therefore the j states are

$$\begin{aligned} |j, m\rangle_{\alpha\beta} &= \frac{(\omega_{1q})^{j+m}}{[j+m]_q!}\frac{(\omega_{2q})^{j+m}}{[j-m]_q!}\exp(\alpha\bar{z}_1 + \beta\bar{z}_2) \\ &= g_{j+m}^\alpha \otimes g'^\beta_{j-m}. \end{aligned} \tag{41}$$

All the properties of the $SU_q(2)$ algebras supplied by [9] and [10] can be verified using the above explicitly constructed representations. But it should be noted that every state $|j,m\rangle$ defined above is related to two arbitrary complex numbers α and β, and therefore every j state is accompanied by a factor $\exp(\alpha\bar{z}_1 + \beta\bar{z}_2)$. This is the main difference between the $SU_q(2)$ symmetry in prequantized and quantum systems.

After the Kähler polarization, we arrive at the quantum system and now the antiholomorphic parts in generators (28) and representation (30) vanish, and therefore $\alpha, \beta \to 0$. The representation in (30) becomes

$$\overline{G} = \{\overline{g}_n, \quad n = 0, 1, 2, \ldots\}$$

$$\overline{g}_n = \frac{z^n}{[n]_q!} = \frac{\hat{\omega}_q^n}{[n]_q!}\overline{g}_0. \tag{42}$$

The representation spaces for $SU_{q,\hbar}(2)$ in (20)–(22) should be the tensor products of the Hilbert spaces of the algebras generated by $\{\hat{\omega}_{i_q}, \hat{\bar{\omega}}_{i_q}, \omega_i\hat{\bar{\omega}}_i\}$ with $i = 1, 2$. Therefore the j states are

$$|j,m\rangle_{\alpha\beta} = \frac{(\hat{\omega}_{1_q})^{j+m}}{[j+m]!}\frac{(\hat{\omega}_{2_q})^{j+m}}{[j-m]!}$$

$$= g_{j+m} \otimes g'_{j-m}\,. \tag{43}$$

4. Remarks and discussions

In this paper we have constructed the Hopf algebraic structures for both $SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$ in terms of the geometric quantization method. The corresponding representations are also discussed in detail. Although we have dealt with the particular algebra of $SU_q(2)$, the principle of the method presented in [17, 18] and this paper can be applied to more general cases more or less straightforwardly.

The Hopf algebraic structure for the $SU_{q\hbar\to 0}(2)$ algebra is of some significance as it makes $SU_{q\hbar\to 0}(2)$ a quantum algebra in the usual sense. Also, provided that

$$J_+J_- + J_3^2 = s_0^2 \tag{44}$$

with s_0 a constant (the total angular momentum), the objects $J'_\pm$ and J'_3 satisfy [18]

$$J'_+J'_- + \frac{(\sinh\gamma J'_3)^2}{\gamma\sinh\gamma} = \frac{(\sinh\gamma s_0)^2}{\gamma\sinh\gamma}. \tag{45}$$

Equation (45) defines a q-deformed sphere of the manifold S_q^2 which is related to the q-deformed Hopf bundle $S^3 \to S_q^2$ and therefore the q-deformed monopole as the manifold S^2 defined by equation (44) is related to the usual Hopf bundle $S^3 \to S^2$ and therefore the monopole [18]. Hence there is also a relation between the oscillator approach of the q-deformation of the SU(2) algebra and the monopole as well as the q-monopole descriptions.

The representations of $SU_{q\hbar\to 0}(2)$ and SU(2) presented in this paper at prequantization level are very intriguing. There should be some physical meaning and applications in classical mechanics, which will be exploited elsewhere.

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